

Twists over étale groupoids and twisted vector bundles

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Abstract

Inspired by recent papers on twisted K -theory, we consider in this article the question of when a twist \mathcal{R} over a locally compact Hausdorff groupoid \mathcal{G} (with unit space a CW-complex) admits a twisted vector bundle, and we relate this question to the Brauer group of \mathcal{G} . We show that the twists which admit twisted vector bundles give rise to a subgroup of the Brauer group of \mathcal{G} . When \mathcal{G} is an étale groupoid, we establish conditions (involving the classifying space $B\mathcal{G}$ of \mathcal{G}) which imply that a torsion twist \mathcal{R} over \mathcal{G} admits a twisted vector bundle.

1 Introduction

C^* -algebras associated to dynamical systems have provided motivation and examples for a wide array of topics in C^* -algebra theory: representation theory, ideal structure, K -theory, classification, and connections with mathematical physics, to name a few. In many of these cases, a complete understanding of the theory has required expanding the notion of a dynamical system to allow for partial actions and twisted actions, as well as actions of group-like objects such as semigroups or groupoids.

For example, the C^* -algebras $C^*(\mathcal{G}; \mathcal{R})$ associated to a groupoid \mathcal{G} and a twist \mathcal{R} over \mathcal{G} (hereafter referred to as *twisted groupoid C^* -algebras*) provide important insights into mathematical physics as well as the structure of other C^* -algebras. First, the collection of twists \mathcal{R} over a groupoid \mathcal{G} is intimately related with the cohomology of \mathcal{G} , cf. [15, 16, 26]. Another structural result is due to Kumjian [14] and Renault [24]: groupoid twists classify Cartan pairs. Finally, the papers [27, 4, 2] establish that twisted groupoid C^* -algebras classify D -brane charges in many flavors of string theory.

We also note, following [22, 27], that groupoid twists constitute an example of Fell bundles. Indeed, Fell bundles provide a universal framework for studying all of the generalized dynamical systems mentioned above.

In several recent papers (cf. [3, 27, 10]) on twisted groupoid C^* -algebras, the K -theory groups of these C^* -algebras have received a good deal of attention. Of particular interest is the question of when $K_0(C^*(\mathcal{G}; \mathcal{R}))$ can be

completely understood in terms of \mathcal{G} -equivariant vector bundles. Phillips established in Chapter 9 of [23] that \mathcal{G} -equivariant vector bundles may not suffice to describe $K_0(C^*(\mathcal{G}; \mathcal{R}))$, even when $\mathcal{G} = M \rtimes G$ is a transformation group and \mathcal{R} is trivial. Vector bundles provide a highly desirable geometric perspective on $K_0(C^*(\mathcal{G}; \mathcal{R}))$, however, and so conditions are sought (cf. [1, 2, 5, 9, 10, 18]) under which $K_0(C^*(\mathcal{G}; \mathcal{R}))$ is generated by \mathcal{G} -equivariant vector bundles.

In Theorem 5.28 of [27], Tu, Xu, and Laurent-Gengoux study this question for proper Lie groupoids \mathcal{G} . They establish, in this context, sufficient conditions for the K -theory group $K_0(C^*(\mathcal{G}, \mathcal{R}))$ associated to a twist \mathcal{R} over \mathcal{G} to be generated by $(\mathcal{R}, \mathcal{G})$ -twisted vector bundles over the unit space of \mathcal{G} (see Definition 2.4 below). A necessary condition is that \mathcal{R} be a torsion element of the Brauer group of \mathcal{G} . Conjecture 5.7 on page 888 of [27] states that, if \mathcal{G} is a proper Lie groupoid acting cocompactly on its unit space, then this condition is also sufficient.

Conjecture 5.7 of [27] has not yet been disproved, but it has only been proven true in certain special cases: cf. [18, 10, 5] when $\mathcal{R} = \mathcal{G} \times \mathbb{T}$ is the trivial twist, [2] for nontrivial twists \mathcal{R} over manifolds M , and [1, 9, 19] for nontrivial twists over representable orbifolds $G \rtimes M$, where G is a discrete group acting properly on a compact space M .

In hopes of shedding more light on this Conjecture, we present an equivalent formulation in Conjecture 3.5 below, using the Brauer group of \mathcal{G} as defined in [16]. Our reformulated conjecture relies on our result (Proposition 3.4) that, for any locally compact Hausdorff groupoid \mathcal{G} whose unit space is a CW-complex, the collection of twists \mathcal{R} over \mathcal{G} which admit twisted vector bundles gives rise to a subgroup $Tw_\tau(\mathcal{G})$ of the Brauer group $Br(\mathcal{G})$.

We note that Theorem 3.2 of [13] also establishes a link between twisted vector bundles and the Brauer group, but Karoubi's approach in [13] differs substantially from ours, and does not address the group structure of $Tw_\tau(\mathcal{G})$.

In the second part of the paper, we address the question of when a torsion twist \mathcal{R} over an étale groupoid \mathcal{G} admits a twisted vector bundle. The existence of such vector bundles is necessary (but not sufficient) in order for $K_0(C^*(\mathcal{G}; \mathcal{R}))$ to be generated by twisted vector bundles.

Theorem 4.6 below establishes that if the classifying space $B\mathcal{G}$ is a compact CW-complex and if a certain principal $PU(n)$ -bundle P lifts to a $U(n)$ -principal bundle \tilde{P} , then up to Morita equivalence, the torsion twist \mathcal{R} admits a twisted vector bundle. To our knowledge, the connection between classifying spaces and twisted vector bundles has not been explored previously in the literature; we are optimistic that Theorem 4.6 will lead to new insights into the K -theory of twisted groupoid C^* -algebras.

1.1 Structure of the paper

We begin in Section 2 by reviewing the basic concepts we will rely on throughout this paper: locally compact Hausdorff groupoids, twists over such groupoids, groupoid vector bundles and twisted vector bundles. In Section 3 we show that, for any locally compact Hausdorff groupoid \mathcal{G} whose unit space is a CW-

complex, the collection of twists over \mathcal{G} which admit twisted vector bundles gives rise to a subgroup of the Brauer group of \mathcal{G} , and we use this to present an alternate formulation of Conjecture 5.7 from [27]. Finally in Section 4 we consider torsion twists for étale groupoids. We establish, in Theorem 4.6, sufficient conditions for a torsion twist \mathcal{R} over an étale groupoid \mathcal{G} to admit a twisted vector bundle, and we present examples showing that the hypotheses of Theorem 4.6 are satisfied in many cases of interest.

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2 Definitions

Recall that a *groupoid* is a small category with inverses. Throughout this note, \mathcal{G} will denote (the space of arrows of) a groupoid with unit space $\mathcal{G}^{(0)}$, with source, range (or target), and unit maps

$$s, r : \mathcal{G} \longrightarrow \mathcal{G}^{(0)}, \quad u : \mathcal{G}^{(0)} \longrightarrow \mathcal{G}.$$

As usual we denote the set of composable elements of \mathcal{G} by $\mathcal{G}^{(2)}$, where

$$\mathcal{G}^{(2)} = \mathcal{G} \times_{s, \mathcal{G}^{(0)}, t} \mathcal{G} = \{(g_1, g_2) \in \mathcal{G} \times \mathcal{G} \mid s(g_1) = r(g_2)\}.$$

In this paper we will primarily be concerned with *locally compact Hausdorff groupoids*. These are groupoids \mathcal{G} such that the spaces $\mathcal{G}^{(0)}, \mathcal{G}, \mathcal{G}^{(2)}$ have locally compact Hausdorff topologies with respect to which the maps $s, r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$, the multiplication $\mathcal{G}^{(2)} \rightarrow \mathcal{G}$, and the inverse map $\mathcal{G} \rightarrow \mathcal{G}$ are continuous. Conjecture 3.5 below makes reference to *Lie groupoids*, which are locally compact Hausdorff groupoids such that the spaces $\mathcal{G}^{(0)}, \mathcal{G}, \mathcal{G}^{(2)}$ are smooth manifolds and all of the structure maps between them are smooth.

Theorem 4.6 deals with *étale groupoids*, which are locally compact Hausdorff groupoids \mathcal{G} for which r, s are local homeomorphisms. For example, if a discrete group Γ acts on a CW-complex M , the associated transformation group $\Gamma \ltimes M$ is an étale groupoid.

Definition 2.1. Let $\mathcal{G}_1, \mathcal{G}_2$ be two locally compact Hausdorff groupoids with unit spaces $\mathcal{G}_1^{(0)}, \mathcal{G}_2^{(0)}$ respectively. A morphism $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ consists of a pair of continuous maps $f = (f_0, f_1)$, with

$$f_0 : \mathcal{G}_1^{(0)} \rightarrow \mathcal{G}_2^{(0)}, \quad f_1 : \mathcal{G}_1 \rightarrow \mathcal{G}_2,$$

such that, if we denote by $s_{\mathcal{G}_j}$ and $r_{\mathcal{G}_j}$ the source and range maps of \mathcal{G}_j , $j = 1, 2$, we have

$$s_{\mathcal{G}_2} \circ f_1 = f_0 \circ s_{\mathcal{G}_1}, \quad \text{and} \quad r_{\mathcal{G}_2} \circ f_1 = f_0 \circ r_{\mathcal{G}_1}.$$

The notion of a twist or \mathbb{T} -central extension of a groupoid \mathcal{G} was originally developed (cf. [14, 21, 27]) to provide a “second cohomology group” for groupoids. Groupoid twists and their associated twisted vector bundles (see Definition 2.4 below) are the groupoid analogues of group 2-cocycles and projective representations.

Definition 2.2. Let \mathcal{G} be a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$. A \mathbb{T} -central extension (or “twist”) of \mathcal{G} consists of

1. A locally compact Hausdorff groupoid \mathcal{R} with unit space $\mathcal{G}^{(0)}$, together with a morphism of locally compact Hausdorff groupoids

$$(id, \pi) : \mathcal{R} \rightarrow \mathcal{G}$$

which restricts to the identity on $\mathcal{G}^{(0)}$.

2. A left \mathbb{T} -action on \mathcal{R} , with respect to which \mathcal{R} is a left principal \mathbb{T} -bundle over \mathcal{G} .
3. These two structures are compatible in the sense that

$$(z_1 r_1)(z_2 r_2) = z_1 z_2 (r_1 r_2), \forall z_1, z_2 \in \mathbb{T}, \forall (r_1, r_2) \in \mathcal{R}^{(2)} = \mathcal{R} \times_{s, \mathcal{G}^{(0)}, r} \mathcal{R}.$$

We write $Tw(\mathcal{G})$ for the set of twists over \mathcal{G} .

These conditions (1)-(3) imply the exactness of the sequence of groupoids

$$\mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(0)} \times \mathbb{T} \rightarrow \mathcal{R} \rightarrow \mathcal{G} \rightrightarrows \mathcal{G}^{(0)},$$

which highlights the parallel between twists over a groupoid \mathcal{G} and extensions of \mathcal{G} by \mathbb{T} (or elements of the second cohomology group $H^2(\mathcal{G}, \mathbb{T})$).

If $\mathcal{R}_1, \mathcal{R}_2 \in Tw(\mathcal{G})$, we can form their Baer sum

$$\mathcal{R}_1 + \mathcal{R}_2 := \{(r_1, r_2) \in \mathcal{R}_1 \times \mathcal{R}_2 : \pi_1(r_1) = \pi_2(r_2)\} / \sim,$$

where $(r_1, r_2) \sim (zr_1, \bar{z}r_2)$ for all $z \in \mathbb{T}$. Define an action of \mathbb{T} on $\mathcal{R}_1 + \mathcal{R}_2$ by $z \cdot [(r_1, r_2)] = [(zr_1, r_2)] = [(r_1, zr_2)]$, and observe that with this action, $\mathcal{R}_1 + \mathcal{R}_2$ becomes a twist over \mathcal{G} .

With this operation $Tw(\mathcal{G})$ becomes a group; the identity element is the trivial extension $\mathcal{G} \times \mathbb{T}$, and the inverse of a twist \mathcal{R} is the twist $\overline{\mathcal{R}}$. As groupoids, $\mathcal{R} = \overline{\mathcal{R}}$; however, the action of \mathbb{T} on $\overline{\mathcal{R}}$ is the conjugate of the action on \mathcal{R} . To be precise, if $r \in \mathcal{R}$, denote by \bar{r} the corresponding element of $\overline{\mathcal{R}}$. Then

$$z \cdot \bar{r} = \overline{\bar{z} \cdot r}.$$

In this note, we will consider actions of groupoids \mathcal{G} and twists \mathcal{R} over \mathcal{G} on a variety of spaces. We make this concept precise as follows.

Definition 2.3. Let \mathcal{G} be a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$. A \mathcal{G} -space is a locally trivial fiber bundle $J : P \rightarrow \mathcal{G}^{(0)}$ such that, setting

$$\mathcal{G} * P = \{(g, p) \in \mathcal{G} \times P : s(g) = J(p)\}$$

and equipping $\mathcal{G} * P$ with the subspace topology inherited from $\mathcal{G} \times P$, we have a continuous map $\sigma : \mathcal{G} * P \rightarrow P$ satisfying

- $\sigma(J(p), p) = p$ for all $p \in P$;
- $J(\sigma(g, p)) = r(g)$ for all $(g, p) \in \mathcal{G} * P$;
- If $(g, h) \in \mathcal{G}^{(2)}$ and $(h, p) \in \mathcal{G} * P$, then $\sigma(g, \sigma(h, p)) = \sigma(gh, p)$.

We will often write $g \cdot p$ for $\sigma(g, p) \in P$.

Note that, as a consequence of the above definition, the map $\sigma_g : P_{s(g)} \rightarrow P_{r(g)}$ given by $p \mapsto \sigma(g, p)$ must be a homeomorphism, for all $g \in \mathcal{G}$.

Definition 2.4. 1. Let \mathcal{G} be a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$, where $\mathcal{G}^{(0)}$ is a CW-complex. A \mathcal{G} -vector bundle is a vector bundle $J : E \rightarrow \mathcal{G}^{(0)}$ which is a \mathcal{G} -space in the sense of Definition 2.3.

2. Let

$$\mathcal{G}^{(0)} \rightarrow \mathbb{T} \times \mathcal{G}^{(0)} \xrightarrow{i} \mathcal{R} \xrightarrow{j} \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$$

be a \mathbb{T} -central extension of locally compact Hausdorff groupoids. By a $(\mathcal{G}, \mathcal{R})$ -twisted vector bundle, we mean a \mathcal{R} -vector bundle $J : E \rightarrow \mathcal{G}^{(0)}$ such that, whenever $z \in \mathbb{T}$, $r \in \mathcal{R}$, $e \in E$ such that $s(r) = J(e)$, we have

$$(z \cdot r) \cdot e = z(r \cdot e). \quad (1)$$

Here, the action on the right-hand side of the equation is simply scalar multiplication (identifying \mathbb{T} with the unit circle of \mathbb{C}).

3. An equivalent characterization of $(\mathcal{R}, \mathcal{G})$ -twisted vector bundles is the following:

A \mathcal{R} -vector bundle $E \rightarrow \mathcal{G}^{(0)}$ is a $(\mathcal{R}, \mathcal{G})$ -twisted vector bundle if and only if the subgroupoid $\ker j \cong \mathcal{G}^{(0)} \times \mathbb{T}$ of \mathcal{R} acts on E by scalar multiplication, where \mathbb{T} is identified with the unit circle of \mathbb{C} .

In Proposition 3.4, we will establish a connection between the twists over \mathcal{G} which admit twisted vector bundles and the Brauer group of \mathcal{G} , as introduced in [16]. Thus, we review here a few facts about the Brauer group and its connection to $Tw(\mathcal{G})$.

Definition 2.5. Let \mathcal{G} be a locally compact Hausdorff groupoid. As in Definition 8.1 of [16], we will denote by $Br_0(\mathcal{G})$ the group of Morita equivalence classes of \mathcal{G} -spaces \mathcal{A} such that $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{K}(\mathcal{H})$ for some Hilbert space \mathcal{H} . We denote the class in $Br_0(\mathcal{G})$ of \mathcal{A} by $[\mathcal{A}, \alpha]$, where α is the action of \mathcal{G} on \mathcal{A} .

Also, let $\mathcal{E}(\mathcal{G})$ be the quotient of $Tw(\mathcal{G})$ by Morita equivalence or, equivalently, the quotient by the subgroup W of elements which are Morita equivalent to the trivial twist. See Definition 3.1 and Corollary 7.3 of [16] for details.

Theorem 8.3 of [16] establishes that

$$Br_0(\mathcal{G}) \cong \mathcal{E}(\mathcal{G}) = Tw(\mathcal{G})/W.$$

3 Twisted vector bundles and the Brauer group

Let \mathcal{G} be a locally compact Hausdorff groupoid with unit space a CW-complex. In this section, we will show that the subset $Tw_\tau(\mathcal{G})$ of twists over \mathcal{G} which admit twisted vector bundles gives a subgroup of $Br_0(\mathcal{G})$.

Definition 3.1. For a locally compact Hausdorff groupoid \mathcal{G} , let $Br_\tau(\mathcal{G})$ be the subgroup of $Br_0(\mathcal{G})$ consisting of Morita equivalence classes $[\mathcal{A}, \alpha]$ of elementary \mathcal{G} -bundles $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{K}(\mathcal{H})$ with zero Dixmier-Douady invariant, such that \mathcal{H} is finite dimensional.

When, in addition, the unit space of \mathcal{G} is a CW-complex, we denote by $Tw_\tau(\mathcal{G})$ the subset of $Tw(\mathcal{G})$ consisting of twists \mathcal{R} over \mathcal{G} that admit a twisted vector bundle.

Proposition 3.2. *Let \mathcal{G} be a locally compact groupoid whose unit space is a CW-complex. Then $Tw_\tau(\mathcal{G})$ is a subgroup of $Tw(\mathcal{G})$.*

Proof. 1. (Closure under operation) Given two twists

$$\mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i_1} \mathcal{R}_1 \xrightarrow{j_1} \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}, \quad \mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i_2} \mathcal{R}_2 \xrightarrow{j_2} \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$$

that admit twisted vector bundles E_1 and E_2 respectively, it is straightforward to show that

$$E_1 * E_2 := \{(e_1, e_2) \in E_1 \oplus E_2 \mid J_1(e_1) = J_2(e_2)\} / \sim,$$

is a twisted vector bundle for the Baer sum $\mathcal{R}_1 + \mathcal{R}_2$. The action of $\mathcal{R}_1 + \mathcal{R}_2$ on $E_1 * E_2$ is given by

$$[(r_1, r_2)] \cdot [(e_1, e_2)] = [(r_1 \cdot e_1, r_2 \cdot e_2)].$$

2. (Neutral Element) The neutral element of $Tw(\mathcal{G})$ is $\mathcal{G} \times \mathbb{T}$. Note that $\mathcal{G} \times \mathbb{T}$ admits a twisted vector bundle E – namely, $E = \mathcal{G}^{(0)} \times \mathbb{C}$, with the action $(g, z) \cdot (s(g), v) = (r(g), zv)$.
3. (Inverses) We must show that, if $\mathcal{R} \in Tw_\tau(\mathcal{G})$, then $\overline{\mathcal{R}} \in Tw_\tau(\mathcal{G})$.

For $\mathcal{R} \in Tw_\tau(\mathcal{G})$, let $E \rightarrow \mathcal{G}^{(0)}$ be a $(\mathcal{R}, \mathcal{G})$ -twisted vector bundle. Write \overline{E} for the conjugate vector bundle – that is, $\overline{E} = E$ as sets, and the additive operation on \overline{E} agrees with that on E (in symbols, $e + f = \overline{e} + \overline{f}$), but the \mathbb{C} action on \overline{E} is the conjugate of the action on E : $z \cdot \overline{e} = \overline{z} \cdot e$. Define an action of $\overline{\mathcal{R}}$ on \overline{E} by $\overline{r} \cdot \overline{e} = \overline{r \cdot e}$. This action makes \overline{E} into a $\overline{\mathcal{R}}$ -vector bundle since E is a \mathcal{R} -vector bundle. Moreover, for any $z \in \mathbb{T}$ we have

$$\begin{aligned} (z \cdot \overline{r}) \cdot \overline{e} &= \overline{\overline{z} \cdot r} \cdot \overline{e} = \overline{(\overline{z}r) \cdot e} \\ &= \overline{\overline{z}(r \cdot e)} = z \overline{r \cdot e} \\ &= z(\overline{r} \cdot \overline{e}). \end{aligned}$$

Thus, \mathbb{T} acts by scalars on \overline{E} , and so \overline{E} is a $(\overline{W}, \mathcal{G})$ -twisted vector bundle. \square

Remark 3.3. Recall from Proposition 5.5 of [27] that if a twist \mathcal{R} over \mathcal{G} admits a twisted vector bundle, then \mathcal{R} must be torsion. Thus, $Tw_\tau(\mathcal{G})$ is a subgroup of $Tw^{tor}(\mathcal{G})$, the torsion subgroup of $Tw(\mathcal{G})$.

3.1 The image of $Tw_\tau(\mathcal{G})$ in $Br_0(\mathcal{G})$

Recall that if \mathcal{G} is a locally compact Hausdorff groupoid with unit space $\mathcal{G}^{(0)}$, then $Br_0(\mathcal{G})$ consists of Morita equivalence classes of \mathcal{G} -spaces of the form $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{K}(\mathcal{H})$.

In Section 8 of [16], the authors construct an isomorphism $\Theta : Br_0(\mathcal{G}) \rightarrow Tw(\mathcal{G})/W$, where W is the subgroup of $Tw(\mathcal{G})$ consisting of elements which are Morita equivalent to the trivial twist. We will use this isomorphism to study the subgroup of $Br_0(\mathcal{G})$ corresponding to $Tw_\tau(\mathcal{G})$.

Proposition 8.7 of [16] describes a homomorphism $\theta : Tw(\mathcal{G}) \rightarrow Br_0(\mathcal{G})$ which induces the inverse of Θ .

Proposition 3.4. *Suppose \mathcal{G} is a locally compact Hausdorff groupoid whose unit space $\mathcal{G}^{(0)}$ is a connected CW-complex, and suppose $\mathcal{R} \in Tw_\tau(\mathcal{G})$. Then there exists a finite-dimensional \mathcal{G} -vector bundle $V \rightarrow \mathcal{G}^{(0)}$ such that $\theta(\mathcal{R}) = [\text{Aut}(V), \alpha]$, where α is induced by the action of \mathcal{G} on V .*

Moreover, if $[\mathcal{A}, \alpha'] \in Br_0(\mathcal{G})$ and (\mathcal{A}, α') is Morita equivalent to (\mathcal{M}_n, α) where \mathcal{M}_n is an $M_n(\mathbb{C})$ -bundle over $\mathcal{G}^{(0)}$, then $[\mathcal{A}, \alpha'] = [\mathcal{M}_n, \alpha]$ lies in $\theta(Tw_\tau(\mathcal{G}))$.

In other words, $Br_\tau(\mathcal{G}) \cong Tw_\tau(\mathcal{G})$.

Proof. If $\mathcal{R} \in Tw_\tau(\mathcal{G})$ and V is a $(\mathcal{R}, \mathcal{G})$ -twisted vector bundle, write $j : \mathcal{R} \rightarrow \mathcal{G}$ for the projection of \mathcal{R} onto \mathcal{G} and write $\sigma : \mathcal{R} * V \rightarrow V$ for the action of \mathcal{R} on V . Define $\alpha : \mathcal{G} * \text{Aut}(V) \rightarrow \text{Aut}(V)$ by

$$(\alpha(g, A)(v) = \sigma(\eta, A(\sigma(\eta^{-1}, v))),$$

where $v \in V_{\tau(g)}$ and $\eta \in j^{-1}(g)$.

Note that $\alpha(g, A)$ does not depend on our choice of $\eta \in j^{-1}(g)$: If $\eta, \eta' \in j^{-1}(g)$, the fact that \mathcal{R} is a principal \mathbb{T} -bundle over \mathcal{G} implies that $\eta = z\eta'$ for some $z \in \mathbb{T}$. Since V is a $(\mathcal{G}, \mathcal{R})$ -twisted vector bundle, $\sigma(\eta, v) = z\sigma(\eta', v)$, and consequently

$$\sigma(\eta, A(\sigma(\eta^{-1}, v))) = \sigma(\eta', A(\sigma((\eta')^{-1}, v))).$$

Now, Lemma 8.8 of [16] establishes that $[\text{Aut}(V), \alpha] = \theta(\mathcal{R})$.

For the second statement, suppose α is an action of \mathcal{G} on a bundle \mathcal{M}_n of n -dimensional matrix algebras over $\mathcal{G}^{(0)}$. Then Theorem 8.3 of [16] explains how to construct the twist $\Theta([\mathcal{M}_n, \alpha])$, using a pullback construction. To be precise,

$$\Theta([\mathcal{M}_n, \alpha]) = \{(g, U) \in \mathcal{G} \times U_n(\mathbb{C}) : \alpha_g = \text{Ad } U\} =: \mathcal{R}(\alpha).$$

We will construct a $(\mathcal{G}, \mathcal{R}(\alpha))$ -twisted vector bundle, proving that $[\mathcal{M}_n, \alpha] \in \theta(Tw_\tau(\mathcal{G}))$.

The \mathbb{T} -action on $\mathcal{R}(\alpha)$ which makes it into a twist over \mathcal{G} is given by

$$z \cdot (g, U) = (g, z \cdot U).$$

Consider the sub-bundle \mathcal{GL}_n of \mathcal{M}_n obtained by considering only the invertible elements of $M_n(\mathbb{C})$ in each fiber of \mathcal{M}_n . Notice that $GL_n(\mathbb{C})$ acts on \mathcal{GL}_n by right multiplication in each fiber, and that this action is continuous, and free and transitive in each fiber, and hence makes \mathcal{GL}_n into a principal GL_n bundle. We consequently obtain an associated vector bundle over $\mathcal{G}^{(0)}$,

$$V = \mathcal{GL}_n \times_{GL_n(\mathbb{C})} \mathbb{C}^n.$$

Moreover, $\mathcal{R}(\alpha)$ acts on V :

$$(g, U) \cdot [A, v] = [\alpha_g(A), Uv].$$

To see that this action is well defined, take $G \in GL_n(\mathbb{C})$ and calculate:

$$\begin{aligned} [\alpha_g(AG), U(G^{-1}v)] &= [UAGU^{-1}, UG^{-1}v] = [UA, v] = [UAU^{-1}, Uv] \\ [\alpha_g(A), v] &= [UAU^{-1}, Uv]. \end{aligned}$$

Moreover,

$$\begin{aligned} (z \cdot (g, U)) \cdot [A, v] &= [\alpha_g(A), zU(v)] \\ &= z \cdot [\alpha_g(A), Uv] = z \cdot ((g, U) \cdot [A, v]), \end{aligned}$$

so V is an $(\mathcal{R}(\alpha), \mathcal{G})$ -twisted vector bundle. Thus, $\mathcal{R}(\alpha) \in Tw_\tau(\mathcal{G})$ whenever $[\alpha, \mathcal{M}] \in Br_0(\mathcal{G})$. \square

Proposition 3.4 thus establishes that twists \mathcal{R} over \mathcal{G} which admit twisted vector bundles correspond to C^* -bundles over $\mathcal{G}^{(0)}$ with finite-dimensional fibers. Phrased in this way, the parallel between Proposition 3.4 and Theorem 3.2 of [13] becomes evident. However, the two proofs take very different approaches. Moreover, Karoubi does not address the group structure of $Tw_\tau(\mathcal{G})$ in Theorem 3.2 of [13].

Proposition 3.4 also allows us to rephrase Conjecture 5.7 of [27] in terms of the Brauer group, as follows. Recall that, in its original form, Conjecture 5.7 of [27] asserts that all torsion elements of $Tw(\mathcal{G})$ should admit twisted vector bundles, if \mathcal{G} is proper and the quotient $\mathcal{G}^{(0)}/\mathcal{G}$ is compact.

Conjecture 3.5 ([27] Conjecture 5.7). Let \mathcal{G} be a proper Lie groupoid such that the quotient $\mathcal{G}^{(0)}/\mathcal{G}$ is compact, and let $[\mathcal{A}, \alpha] \in Br_0(\mathcal{G})$ be a torsion element. Then $[\mathcal{A}, \alpha] = [\mathcal{M}, \alpha']$ for some finite-dimensional matrix algebra bundle \mathcal{M} over $\mathcal{G}^{(0)}$ and an action α' of \mathcal{G} on \mathcal{M} .

4 Twisted vector bundles for étale groupoids

In this section we consider torsion twists over étale groupoids \mathcal{G} . We establish in Theorem 4.6 sufficient conditions for a torsion twist \mathcal{R} over \mathcal{G} to admit (up to Morita equivalence) a twisted vector bundle, and we describe examples meeting these conditions in Section 4.1. The conditions of Theorem 4.6 are phrased in terms of the classifying space $B\mathcal{G}$ and in terms of a principal bundle P associated to \mathcal{R} . Using $B\mathcal{G}$ to study twisted vector bundles appears to be a new approach; this perspective was inspired by Moerdijk's result in [20] identifying $H^*(\mathcal{G}, \mathcal{S})$ and $H^*(B\mathcal{G}, \tilde{\mathcal{S}})$ for an abelian \mathcal{G} -sheaf \mathcal{S} , and the Serre-Grothendieck Theorem (cf. Theorem 1.6 of [11]) relating $H^1(M, PU(n))$ and $H^2(M, \mathbb{T})$ for M a CW-complex.

We begin with some preliminary definitions and results.

Definition 4.1. Let \mathcal{G} be a topological groupoid. The *simplicial space associated to \mathcal{G}* is

$$\mathcal{G}_\bullet = \{\mathcal{G}^{(k)}, \epsilon_j^k, \eta_k^j\}_{0 \leq j \leq k \in \mathbb{N}},$$

where $\mathcal{G}^{(k)}$ is the space of composable n -tuples in \mathcal{G} , $\epsilon_j^k : \mathcal{G}^{(k)} \rightarrow \mathcal{G}^{(k-1)}$, and $\eta_k^j : \mathcal{G}^{(k)} \rightarrow \mathcal{G}^{(k+1)}$ are given as follows:

$$\begin{aligned} \epsilon_0^k(g_1, \dots, g_k) &= (g_2, \dots, g_k) \\ \epsilon_i^k(g_1, \dots, g_k) &= (g_1, \dots, g_i g_{i+1}, \dots, g_k) \text{ if } 1 \leq i \leq k-1 \\ \epsilon_k^k(g_1, \dots, g_k) &= (g_1, \dots, g_{k-1}) \end{aligned}$$

If $k = 1$, we have $\epsilon_0^1(g) = s(g)$, $\epsilon_1^1(g) = r(g)$.

The degeneracy maps η_i^k are given for $k \geq 1$ by

$$\begin{aligned} \eta_i^k(g_1, \dots, g_k) &= (g_1, \dots, g_i, s(g_i), g_{i+1}, \dots, g_k) \text{ if } i \geq 1; \\ \eta_0^k(g_1, \dots, g_k) &= (r(g_1), g_1, \dots, g_k). \end{aligned}$$

When $k = 0$, the map $\eta_0^0 : \mathcal{G}^{(0)} \rightarrow \mathcal{G}^{(1)}$ is just the standard inclusion of $\mathcal{G}^{(0)}$ into $\mathcal{G}^{(1)} = \mathcal{G}$.

For the definition of a general simplicial space, see e.g. [17] Section 2.1.

Definition 4.2 (cf. [20, 29]). Let \mathcal{G} be a topological groupoid. A *classifying space* $B\mathcal{G}$ for \mathcal{G} is any space which can be realized as a quotient $B\mathcal{G} = EG/\mathcal{G}$ of a weakly contractible space EG by a free action of \mathcal{G} . When we need an explicit model for $B\mathcal{G}$, we will use the geometric realization $|\mathcal{G}_\bullet|$ of the simplicial space associated to \mathcal{G} :

$$B\mathcal{G} = |\mathcal{G}_\bullet| = \left(\bigsqcup_{k \geq 0} \mathcal{G}^{(k)} \times \Delta^k \right) / \sim,$$

where Δ^k denotes the standard k -simplex.¹ The equivalence relation \sim is defined by $(p, \delta_i^{k-1}v) \sim (\epsilon_i^k p, v)$ for $p \in \mathcal{G}^{(k)}$, $v \in \Delta^{k-1}$, where $\delta_i^{k-1} : \Delta^{k-1} \rightarrow \Delta^k$ is the i th degeneracy map, gluing Δ^{k-1} to the i th face of Δ^k , and $\epsilon_i^k : \mathcal{G}^{(k)} \rightarrow \mathcal{G}^{(k-1)}$ is the i th face map. In other words, we have $\delta_0^0(\emptyset) = 0$, $\delta_1^0(\emptyset) = 1$, and if $k > 1$

$$\delta_i^{k-1}(t_1, \dots, t_{k-1}) = \begin{cases} (0, t_1, \dots, t_{k-1}) & \text{if } i = 0 \\ (t_1, \dots, t_i, t_i, t_{i+1}, \dots, t_k) & \text{if } 1 \leq i \leq k-1 \\ (t_1, \dots, t_{k-1}, 1) & \text{if } i = k. \end{cases}$$

The topology on this model of $B\mathcal{G}$ is the inductive limit topology induced by the natural topologies on $\mathcal{G}^{(n)}$, Δ^n .

Definition 4.3 ([17] Definition 2.2). Let X_\bullet be a simplicial space and let G be a topological group. A *principal G -bundle over X_\bullet* is a simplicial space P_\bullet such that, for each $k \geq 0$, P_k is a principal G -bundle over X_k , and the face and degeneracy maps in P_\bullet are morphisms of principal bundles.

Remark 4.4. Combining [27] Definition 2.1 and Proposition 2.4 of [17], we see that principal G -bundles over \mathcal{G}_\bullet are equivalent to generalized morphisms $\mathcal{G} \rightarrow G$.

Proposition 4.5. *Let \mathcal{G} be an étale groupoid. Suppose that the classifying space $B\mathcal{G}$ is (homotopy equivalent to) a compact CW complex. If $\mathcal{R} \rightarrow \mathcal{G}$ is a twist of order n , then \mathcal{R} gives rise to a principal $PU(n)$ -bundle $P \rightarrow \mathcal{G}^{(0)}$. Moreover, P admits a left action of \mathcal{G} which commutes with the right action of $PU(n)$.*

Proof. For any étale groupoid \mathcal{G} , and any twist $\mathcal{R} \rightarrow \mathcal{G}$, Proposition 11.3, Corollary 7.3, and Theorem 8.3 of [16] combine to tell us that \mathcal{R} determines an element of $H^2(\mathcal{G}, \mathcal{S}^1)$, where \mathcal{S}^1 denotes the sheaf of circle-valued functions on $\mathcal{G}^{(0)}$. The main Theorem of [20] tells us that we then obtain an associated element $[\mathcal{R}]$ of $H^2(B\mathcal{G}, \mathcal{S}^1) \cong H^3(B\mathcal{G}, \mathbb{Z})$. All of the maps $\text{Tw}(\mathcal{G}) \rightarrow H^2(\mathcal{G}, \mathcal{S}^1) \cong H^3(B\mathcal{G}, \mathbb{Z})$ are group homomorphisms, so if \mathcal{R} is a torsion twist of order n , then $n \cdot [\mathcal{R}] = 0$ also in $H^3(B\mathcal{G}, \mathbb{Z})$.

Now, suppose that $B\mathcal{G}$ is a compact CW complex and that \mathcal{R} is a torsion twist of order n . The Serre-Grothendieck theorem (cf. [11] Theorem 1.6, [8] Theorem 8 or [19] Theorem 7.2.11) tells us that \mathcal{R} gives rise to a principal $PU(n)$ bundle Q over $B\mathcal{G}$.

Note that, for each $k \in \mathbb{N}$, the map $\varphi_k : \mathcal{G}^{(k)} \rightarrow B\mathcal{G}$ given by $(g_1, \dots, g_k) \mapsto [(g_1, \dots, g_k), (0, \dots, 0)]$ is continuous. Moreover, the equivalence relation which defines $B\mathcal{G}$ ensures that the maps φ_k commute with the face and degeneracy maps ϵ_i^k, η_i^k :

$$\forall i, \varphi_k \circ \eta_i^{k-1} = \varphi_{k-1} \text{ and } \varphi_{k-1} \circ \epsilon_i^k = \varphi_k.$$

¹For $k > 0$, Δ^k can be realized as a subset of \mathbb{R}^k , namely,

$$\Delta^k = \{(t_1, \dots, t_k) : 0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq 1\}.$$

If $k = 0$, Δ^k consists of one point, and we will denote $\Delta^0 = \emptyset$.

Principal $PU(n)$ -bundles over a space X are classified by homotopy classes of maps $X \rightarrow BPU(n)$, so the maps φ_k allow us to pull back our principal $PU(n)$ -bundle Q over $B\mathcal{G}$ to a principal $PU(n)$ -bundle P_k over $\mathcal{G}^{(k)}$ for each $k \geq 0$. Since the maps φ_k commute with the face and degeneracy maps for \mathcal{G}_\bullet , the maps η_i^k, ϵ_i^k induce morphisms of principal bundles which make P_\bullet into a principal $PU(n)$ -bundle over \mathcal{G}_\bullet in the sense of Definition 4.3. Thus, by Proposition 2.4 of [17], we have a principal $PU(n)$ -bundle P over $\mathcal{G}^{(0)}$ which admits an action of \mathcal{G} . \square

In what follows, we will combine the bundle P_\bullet constructed above with the canonical \mathbb{T} -central extension

$$\beta : 1 \rightarrow \mathbb{T} \rightarrow U(n) \rightarrow PU(n) \xrightarrow{\pi} 1 \quad (2)$$

of $PU(n)$. The Leray spectral sequence for the map $BU(n) \rightarrow PU(n)$ implies that β is a generator of $H^2(PU(n), \mathbb{T}) \cong \mathbb{Z}_n$. When n is prime, an alternate proof of this fact is given in Theorem 3.6 of [28].

These preliminaries completed, we now present the main result of this section.

Theorem 4.6. *Let \mathcal{G} be an étale groupoid. Suppose that the classifying space $B\mathcal{G}$ is (homotopy equivalent to) a compact CW complex. Let $\mathcal{R} \rightarrow \mathcal{G}$ be a twist of order n over \mathcal{G} such that the associated $PU(n)$ -bundle P of Proposition 4.5 lifts to a $U(n)$ -bundle \tilde{P} over $\mathcal{G}^{(0)}$. Then there is a twist \mathcal{T} such that $[\mathcal{T}] = [\mathcal{R}] \in H^2(\mathcal{G}, \mathcal{S}^1)$ and such that \mathcal{T} admits a twisted vector bundle.*

Proof. Recall from [20] that for all $s \in \mathbb{N}$, the inclusion $i : \mathcal{G}_\bullet \rightarrow B\mathcal{G}$ induces an isomorphism $i_s^* : H^s(B\mathcal{G}, \mathbb{T}) \rightarrow H^s(\mathcal{G}, \mathbb{T})$, for all $s \in \mathbb{N}$. Moreover, since i is continuous, it also induces a pullback homomorphism $p_1 : H^1(B\mathcal{G}, PU(n)) \rightarrow H^1(\mathcal{G}_\bullet, PU(n))$, which need not be an isomorphism since $PU(n)$ is not abelian.

Write $v : H^1(B\mathcal{G}, PU(n)) \rightarrow H^2(B\mathcal{G}, \mathbb{T})$ for the Serre map which associates to a principal $PU(n)$ -bundle over $B\mathcal{G}$ its Dixmier-Douady class in $H^2(B\mathcal{G}, \mathbb{T}) \cong H^3(B\mathcal{G}, \mathbb{Z})$. The Serre-Grothendieck Theorem (cf. [8] Theorem 8, [19] Theorem 7.2.11, [11] Theorem 1.6) establishes that

$$v : H^1(B\mathcal{G}, PU(n)) \rightarrow H^3(B\mathcal{G}, \mathbb{Z})$$

is an isomorphism onto the n -torsion subgroup of $H^3(B\mathcal{G}, \mathbb{Z})$ which is induced by the short exact sequence β of Equation (2).

If P is the principal $PU(n)$ -bundle over \mathcal{G} which is associated to \mathcal{R} by Proposition 4.5, examining the constructions employed in the proof of Proposition 4.5 reveals that

$$P = p_1 \circ v^{-1} \circ (i_2^*)^{-1}(\mathcal{R}).$$

Recall from page 860 of [27] that we have a natural map

$$\Phi : H^1(\mathcal{G}_\bullet, PU(n)) \times H^2(PU(n), \mathcal{S}^1) \rightarrow H^2(\mathcal{G}, \mathcal{S}^1),$$

which arises from pulling back a principal $PU(n)$ -bundle over \mathcal{G} along a \mathbb{T} -central extension of $PU(n)$. We claim that

$$\Phi(P_\bullet, \beta) = [\mathcal{R}]. \quad (3)$$

Since Φ is natural, and taking pullbacks preserves cohomology classes, (3) holds because v is induced by β , and β generates $H^2(PU(n), \mathcal{S}^1)$.

We will now use the hypothesis that P admits a lift to a principal $U(n)$ -bundle $\tilde{P} \rightarrow \mathcal{G}^{(0)}$ to show that $\Phi(P_\bullet, \beta)$ is represented by a twist \mathcal{T} which admits a twisted vector bundle. As explained in [27] pp. 860-1, this hypothesis allows us to construct an explicit representative \mathcal{T} of $\Phi(P_\bullet, \beta)$ as follows.

By hypothesis, the quotient map $\pi : U(n) \rightarrow PU(n)$ induces a bundle morphism $\tilde{\pi} : \tilde{P} \rightarrow P$. Write $\frac{P \times P}{PU(n)}$ for the gauge groupoid of the bundle P , and notice that, if $\rho : P \rightarrow \mathcal{G}^{(0)}$ is the projection map of the principal bundle P , we can define a morphism $\varphi : \mathcal{G} \rightarrow \frac{P \times P}{PU(n)}$ as follows. Given $g \in \mathcal{G}$, choose $p \in P$ with $\rho(p) = s(g)$, and define

$$\varphi(g) = [g \cdot p, p].$$

The fact that P is a principal $PU(n)$ -bundle implies that $\varphi(g)$ is a well defined groupoid homomorphism.

We define the twist \mathcal{T} over \mathcal{G} by

$$\mathcal{T} = \{([q_1, q_2], g) \in \frac{\tilde{P} \times \tilde{P}}{U(n)} \times \mathcal{G} : [\tilde{\pi}(q_1), \tilde{\pi}(q_2)] = \varphi(g)\}.$$

We observe that

$$([q_1, q_2], g) \in \mathcal{T} \Leftrightarrow g \cdot \tilde{\pi}(q_2) = \tilde{\pi}(q_1).$$

The backward implication is evident; for the forward implication, note that

$$([q_1, q_2], g) \in \mathcal{T} \Rightarrow \tilde{\pi}(q_2) \in P_{s(g)} \Rightarrow \varphi(g) = [g \cdot \tilde{\pi}(q_2), \tilde{\pi}(q_2)].$$

But also, $([q_1, q_2], g) \in \mathcal{T} \Rightarrow \varphi(g) = [\tilde{\pi}(q_1), \tilde{\pi}(q_2)]$. Note that

$$[\tilde{\pi}(q_1), \tilde{\pi}(q_2)] = [p^1, p^2] \Leftrightarrow \exists u \in PU(n) \text{ s.t. } \tilde{\pi}(q_i) = p^i \cdot u \ \forall i;$$

consequently, $g \cdot \tilde{\pi}(q_2) = \tilde{\pi}(q_1)$ as claimed.

The groupoid structure on \mathcal{T} is given by

$$s([q_1, q_2], g) = s(g), \quad r([q_1, q_2], g) = r(g);$$

if $s(g) = r(h)$ then we define the multiplication by

$$([q_1, q_2], g)([p_1, p_2], h) = ([q_1 \cdot u, p_2], gh),$$

where $u \in U(n)$ is the unique element such that $q_2 \cdot u = p_1 \in \tilde{P}$.

Proposition 2.36 of [27] establishes that \mathcal{T} is a twist over \mathcal{G} such that $[\mathcal{T}] = \Phi(P_\bullet, \beta)$. The action of \mathbb{T} on \mathcal{T} is given by

$$z \cdot ([q_1, q_2], g) = ([q_1 \cdot z, q_2], g). \quad (4)$$

By construction, \mathcal{T} admits a generalized homomorphism $\mathcal{T} \rightarrow U(n)$ which is \mathbb{T} -equivariant. To be precise, the bundle \tilde{P} admits a left action of \mathcal{T} : if $\tilde{p} \in \tilde{P}$ lies in the fiber over $s(g)$, and $([q_1, q_2], g) \in \mathcal{T}$, there exists a unique $u \in U(n)$ such that $q_2 \cdot u = \tilde{p}$. Thus, we define

$$([q_1, q_2], g) \cdot \tilde{p} = q_1 \cdot u.$$

One checks immediately that this action is continuous, \mathbb{T} -equivariant, and commutes with the right action of $U(n)$ on \tilde{P} . In other words, the bundle \tilde{P} equipped with this action constitutes a \mathbb{T} -equivariant generalized morphism $\mathcal{T} \rightarrow U(n)$. Thus, Proposition 5.5 of [27] explains how to construct a $(\mathcal{G}, \mathcal{T})$ -twisted vector bundle. Since $[\mathcal{T}] = \Phi(P_\bullet, \beta) = [\mathcal{R}]$, this completes the proof. \square

4.1 Examples

In this section, we present some examples establishing that the hypotheses of Theorem 4.6 are satisfied in many cases of interest.

Example 4.7. Let M be a compact CW complex, and let α be a homeomorphism of M . If we set $\mathcal{G} = M \rtimes_\alpha \mathbb{Z}$, the first paragraph of [29] Section 1.4.3 tells us that $B\mathcal{G} = M \times_{\mathbb{Z}} \mathbb{R}$. Since M is compact, so is $B\mathcal{G}$.

Example 4.8. (cf. [25] p. 273) Let \mathcal{F} be a foliation of a manifold M . The holonomy groupoid $\mathcal{H}_{\mathcal{F}}$ of \mathcal{F} is an étale groupoid; moreover, if the leaves of the foliation all have contractible holonomy coverings, $B\mathcal{H}_{\mathcal{F}} = M$. Examples of such foliations include the Reeb foliation of S^3 and the Kronecker foliation of \mathbb{T}^n .

In particular, if M is compact, any foliation \mathcal{F} of M with contractible leaves has an associated holonomy groupoid $\mathcal{H}_{\mathcal{F}}$ with $B\mathcal{H}_{\mathcal{F}}$ compact.

Example 4.9. Let $M := \mathbb{R}P^2 \times S^4$. We will identify $\mathbb{R}P^2$ with D^1 / \sim , where (in polar coordinates) $D^1 = \{(\rho, \theta) \in \mathbb{R}^2 : 0 \leq \theta < 2\pi, 0 \leq \rho \leq 1\}$ and $(1, \theta) \sim (1, \theta + \pi)$.

Fix $x \in \mathbb{R} \setminus \mathbb{Q}$, and consider the homeomorphism α of $\mathbb{R}P^2 \times S^4$ given by

$$\alpha([\rho, \theta], z) = ([\rho, \theta + (1 - \rho)x], z).$$

Let $\mathcal{G} = M \rtimes_\alpha \mathbb{Z}$. Since M is compact, Example 4.7 tells us that $B\mathcal{G}$ is compact as well.

By the Künneth Theorem, $\mathbb{Z}/2\mathbb{Z} \cong H^2(\mathbb{R}P^2, \mathbb{Z}) \otimes H^0(S^4, \mathbb{Z})$ is a subgroup of $H^2(M, \mathbb{Z}) \cong H^1(M, \mathbb{T})$. The groupoid \mathcal{G} is an example of a Renault-Deaconu groupoid (cf. [6, 7, 12]); thus, by Theorem 2.2 of [7], twists over $\mathcal{G} = M \rtimes_\alpha \mathbb{Z}$ are classified by $H^1(M, \mathbb{T})$. It follows that \mathcal{G} admits nontrivial torsion twists.

The short exact sequence $1 \rightarrow \mathbb{T} \rightarrow U(n) \rightarrow PU(n) \rightarrow 1$ tells us that the obstruction to a principal $PU(n)$ -bundle over M (an element of $H^1(M, PU(n))$) lifting to a principal $U(n)$ -bundle over M lies in $H^2(M, \mathbb{T}) \cong H^3(M, \mathbb{Z})$. However, the Künneth Theorem tells us that

$$H^3(M, \mathbb{Z}) \cong H^3(\mathbb{R}P^2, \mathbb{Z}) \otimes H^0(S^4, \mathbb{Z}) \cong H^3(\mathbb{R}P^2, \mathbb{Z}) = 0.$$

In other words, every principal $PU(n)$ -bundle over M lifts to a principal $U(n)$ -bundle over M , so every torsion twist over $\mathcal{G} = M \times_{\alpha} \mathbb{Z}$ satisfies the hypotheses of Theorem 4.6.

Furthermore, since the action of \mathbb{Z} on M is not proper, this Example lies outside the cases (cf. [9, 27]) where it was previously known that torsion twists admit twisted vector bundles.

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